Mapping Kitaev's quantum double lattice models to Levin and Wen's string-net models

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We exhibit a mapping identifying Kitaev's quantum double lattice models explicitly as a subclass of Levin and Wen's string-net models via a completion of the local Hilbert spaces with auxiliary degrees of freedom. This identification allows one to carry over to these string-net models the representation-theoretic classification of the excitations in quantum double models as well as define them in arbitrary lattices and provides an illustration of the abstract notion of Morita equivalence. The possibility of generalizing the map to broader classes of string nets is considered.

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I. INTRODUCTION

Quantum lattice models governed by local Hamiltonians exhibit a wealth of phases that sometimes escapes the local group analysis of symmetries underlying Landau's paradigm for second-order phase transitions. Topological phases[1] are a remarkable instance of such exotic behavior, where the effective-field theory controlling the long-distance properties is a topological quantum field theory (TQFT).[2] Such phases arise in the fractional quantum-Hall effect (and also in p-wave superconductors) but they also find explicit realization in a number of lattice models, intensely studied both for their theoretical properties and because of the proposal by Kitaev[3] that they would make intrinsically fault-tolerant quantum memories and computers.

Among these lattice models, the class of quantum double (QD) models, corresponding to discrete lattice gauge theories, was introduced as candidates for quantum memories and computers in Ref. 3. The more comprehensive class of string-net (SN) models was defined in Ref. 4. These models have a characteristic form of frustration-free Hamiltonian as a sum of mutually commuting projectors. It is these two classes of models that we will deal with in this paper. On the other hand, interesting models with more intricate Hamiltonians have been proposed: let us mention the extended Hubbard model;[5] Kitaev's honeycomb model,[6] featuring two-body interactions; color codes,[7] and generalized quantum doubles.[8]

The study of topological order brings together many abstract mathematical concepts, from fields ranging from quantum groups to category theory, yet the simplest topological lattice models offer a very direct physical bridge to these concepts. For instance, the representation theory of quasitriangular Hopf algebras governs the classification of excitations in QD models (to be discussed in Sec. II) and fusion categories are the starting point for SN models.

It is desirable to understand the interrelations among different lattice constructions of systems with topological order. As argued in Ref. 4, two-dimensional SN models encompass all *doubled* topological phases and it is understood that the discrete gauge-theory phases described by QD models should be contained in the class of SN models. In Sec. IV, we show how this happens and identify quantum doubles with a subclass of string-net models (a construction, to the best of our knowledge, not so far made explicit in the literature). In more abstract terms, this is an example of Morita equivalence[9] (the origin of this concept can be found in Ref. 10; see, e.g., Ref. 11), whereby the local degrees of freedom in the lattice may be seen as objects in a category, and the physical excitations, equivalent in both cases, correspond to a representation category.

The plan of the paper is as follows: we briefly introduce the quantum double models in Sec. II and the string-net models in Sec. III. The mapping QD → SN is discussed in Sec. IV and in Sec. V we present our conclusions, and a preview of work in progress concerning generalizations of this construction.

II. QUANTUM DOUBLE MODELS

Quantum double lattice models are a direct translation into the lattice setting of gauge theories with a finite, discrete gauge group G.[12,13] They were proposed by Kitaev as quantum memories in Ref. 3 with the purpose of obtaining anyonic excitations capable of universal quantum computation by braiding. Mochon[14] proved that already the model with the smallest non-Abelian gauge group $G=S_3$ is universal in this sense, assuming certain "magic states" can be prepared. While exhibiting rich non-Abelian anyonic excitations, these models share some of the simplicity of their first instance, the toric code[3] (in particular, topological sectors have integral quantum dimensions). For a recent review using group-theoretic language, see Ref. 8.

The quantum double model based on a finite group G, the $D(G)$ model for short, is defined on an *arbitrary* planar lattice Λ, with local degrees of freedom associated with oriented edges e, the local Hilbert space $\mathcal{H}_e$ being $|G|$ dimensional, with $|G|$ the order of G. The orthonormal computational basis is labeled by group elements, $\mathcal{B}_{\mathbf{B}}=\{|g\rangle | g \in G\}$. The Hilbert space $\mathcal{H}_{e^*}$ for the reversed edge e^* is identified with $\mathcal{H}_e$ via the isomorphism $|g\rangle_e \mapsto |g^{-1}\rangle_{e^*}$.

The Hamiltonian is of the form

$$H^{\mathrm{QD}} = -\sum_v A_v^{\mathrm{QD}} - \sum_p B_p^{\mathrm{QD}}, \tag{1}$$

where p runs over the plaquettes of Λ, v over the vertices, and the A_v^{QD} and B_p^{QD} are mutually commuting projectors acting on the edges surrounding each vertex or plaquette.

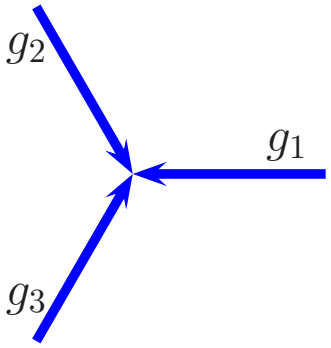

FIG. 1. (Color online) Convention for the definition, Eq. (2), of operators A_v^{QD} in quantum double models. A trivalent vertex is shown but the generalization to arbitrary vertices is straightforward. The operator is symmetric in the different edges.

Their action on computational basis states is defined by the following: for each vertex v, orient its adjacent edges [$\deg(v)$ in number] such that they point toward v. Denote a ket in the computational basis for this set of oriented edges as $|\{g_i\}\rangle = |g_1\rangle \otimes |g_2\rangle \otimes \cdots \otimes |g_{\deg v}\rangle$, where $i=1,\ldots,\deg(v)$ labels the edges (cf. Fig. 1).

Then the vertex projector acts as

$$A_v^{\mathrm{QD}}|\{g_i\}\rangle_v = \frac{1}{|G|}\sum_{h\in G}|\{g_i h^{-1}\}\rangle_v, \tag{2}$$

i.e., it is a simultaneous right multiplication averaged over the group.

For each plaquette p pick an arbitrary adjacent vertex v_0 as starting point and denote $|\{g_j\}\rangle_p$ a ket in the computational basis with $j=1,\ldots,s$ labeling the ordered edges of p along the counterclockwise path starting and ending at v_0 (cf. Fig. 2).

Then the plaquette projector acts as

$$B_p^{\mathrm{QD}}|\{g_j\}\rangle_p = \delta(g_1,\ldots,g_s,e)|\{g_j\}\rangle_p \tag{3}$$

projecting onto states where the product $g_1,\ldots,g_s$ equals the identity element $e\in G$ (δ is a Kronecker delta). Both the choice of the starting vertex v_0 and of the orientation are immaterial but the order is crucial.

Breakdown of any of the ground level constraints $A_v^{\mathrm{QD}}=+1$ and $B_p^{\mathrm{QD}}=+1$ takes the system to excited levels, i.e., to the eigenspaces with eigenvalue -1 of the vertex and plaquette operators. The structure of the excitations can be understood as the appearance of quasiparticles in the regions where constraints are violated. These quasiparticles possess mutual anyonic statistics, which are non Abelian if the group G is non Abelian; they can be classified into superselection sectors (topological charges) given by irreducible representations of the *quantum double* $\mathrm{D}(G)$, a quasitriangular Hopf algebra constructed from the group algebra of the finite group G by Drinfel'd's quantum double construction[15] (a clear introduction for physicists is Ref. 16): magnetic charges, or fluxes, correspond to the violation of a plaquette constraint, $B_p^{\mathrm{QD}}=-1$. They are given by conjugacy classes of elements of G. Electric charges correspond to the violation of a vertex constraint, $A_v^{\mathrm{QD}}=-1$, they are given by irreducible representations (irreps) of G. Dyonic charges correspond to the violation of both a plaquette and a neighboring vertex constraint, they are given by a conjugacy class C of G and an irrep of the centraliser of C in G.

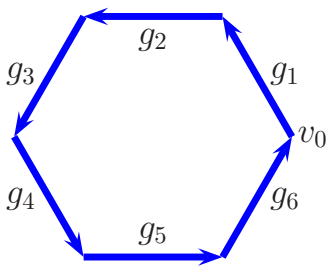

FIG. 2. (Color online) Convention for the definition, Eq. (3), of operators B_p^{QD} in quantum double models. A hexagonal plaquette is shown but the generalization to arbitrary plaquettes is straightforward. The definition of B_p^{QD} does not depend on the choice of v_0 or orientation of the loop.

In addition, each topological sector has a number of internal states (given, for instance, by representatives of a conjugacy class in the case of magnetic charges and by irrep basis vectors in the case of electric charges.) The topological charge can be measured for a region by operations along its boundary and can be defined more generally for any closed loop not necessarily the boundary of a region. Projectors onto the different superselection sectors can be constructed with Kitaev's *ribbon operators*.[3]

III. STRING-NET MODELS

The class of string-net lattice models was introduced in Ref. 4. The intuition behind the construction is that topological lattice Hamiltonians are infrared fixed points of some renormalization-group procedure (this has since been made explicit in Refs. 17 and 18 using entanglement renormalization techniques). String-net models in two dimensions describe all *doubled* topological phases, including interesting cases where the excitations are given by the double semion model or the double Fibonacci models.

The starting point is a honeycomb lattice (or any planar lattice with *trivalent* vertices) with local degrees of freedom along its (oriented) edges. For any oriented edge, an orthonormal basis $\{|a\rangle\}_e$ is labeled by the charges $1,a,b,\ldots,\in M$ in an anyon model (more precisely, a unitary fusion category) featuring particle-antiparticle duality $a\leftrightarrow a^*$, a set of fusion rules $a\times b\rightarrow\Sigma_c N_{ab}^c c$, and fusion/splitting linear spaces such that the recoupling isomorphisms for a $2\rightarrow 2$ process are given by F symbols.[6] The bases for the two different orientations e,e^* of the same edge are related by duality, $|a\rangle_{e^*}=|a^*\rangle_e$.

The string-net Hamiltonian H^{SN} is constructed from the data of the fusion category

$$H^{\mathrm{SN}} = -\sum_v A_v^{\mathrm{SN}} - \sum_p B_p^{\mathrm{SN}}, \tag{4}$$

where v labels the vertices of the lattice and p its plaquettes. Each vertex term A_v^{SN} is a three-body projector (acting on the three vertices incoming to v) favoring fusion rules. That is,

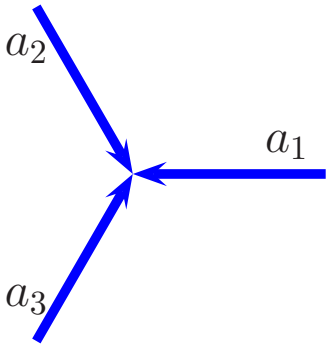

FIG. 3. (Color online) Convention for the definition, Eq. (5), of operators A_v^{SN} in string-net models. The operator is symmetric in the different edges.

they project out vertex configurations $|a,b,c\rangle_v = |a\rangle_{e_1} \otimes |b\rangle_{e_2} \otimes |c\rangle_{e_3}$ with $N_{ab}^{c^*}=0$ (the orientations of the e_j are chosen pointing toward v, see Fig. 3).

We write

$$A_v^{\mathrm{SN}}|a,b,c\rangle_v = \sum_{a,b,c\in M} \delta(a\times b\times c \to 1)|a,b,c\rangle_v \qquad (5)$$

with the obvious meaning for the delta (if $a\times b \to c^* + \cdots$ then $a\times b\times c \to 1 + \cdots$).

Plaquette projectors B_p^{SN} can be constructed from the F symbols of the fusion category. Pictorially, they correspond to the introduction of loops of different labels within the corresponding plaquette and then subsuming the loop into the original lattice by means of recoupling (F) moves (this can be made precise in the fattened lattice picture of the string nets;[4] technically, the loop is to be introduced enclosing the puncture for plaquette p). We denote the operation associated with a loop with label c as $B_p^{\mathrm{SN}}(c)$ and then

$$B_p^{\mathrm{SN}} = \sum_{c\in M} \frac{d_c}{\mathcal{D}^2} B_p^{\mathrm{SN}}(c), \qquad (6)$$

where $\{d_a\}$ are the *quantum dimensions* of the different labels in the fusion category and $\mathcal{D}^2=\Sigma_{b\in M} d_b^2$ is the *total quantum dimension*. The explicit action of the $B_p(a)$'s is spelt out in Ref. 4; the net result for Eq. (6) is

$$B_p^{\mathrm{SN}}|\{a_i\};\{b_j\}\rangle_p = \sum_{c,\{a'\}} \frac{d_c}{\mathcal{D}^2}\left(\prod_{\ell=1}^{6} F_{c a_{\ell-1} a_\ell^*}^{b_j a_\ell'^* a_{\ell-1}'}\right)|\{a_i'\};\{b_j\}\rangle_p, \quad (7)$$

where $\{a_i\}=\{a_1,\ldots,a_6\}$ label the edges along a loop following the boundary of the plaquette (taken hexagonal for definiteness) counterclockwise and $\{b_i\}=\{b_1,\ldots,b_6\}$ is the configuration of the edges immediately neighboring these and pointing toward p (cf. Fig. 4).

The action of B_p^{SN} depends on the b's but only changes the a labels; all in all, a twelve-body operator. All plaquette and vertex constraints can be seen to commute with each other. This allows for a quite explicit treatment of the models. Levin and Wen studied the properties of physical excitations (which constitute a complete anyon model or a unitary *braided* tensor category in the language of Ref. 6) by looking at loop operators commuting with the Hamiltonian. However, there is no general representation-theoretic classification of excitations as that for quantum double models (cf. Sec. II).

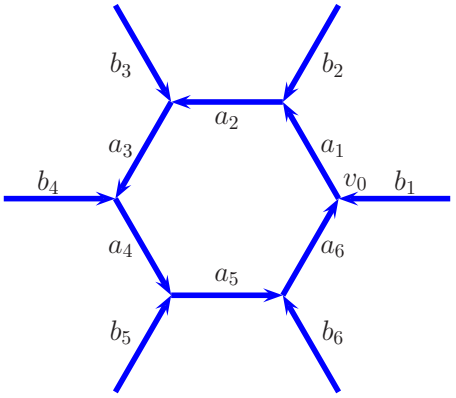

FIG. 4. (Color online) Convention for the definition, Eq. (7), of operators B_p^{SN} in string-net models. The definition of B_p^{SN} does not depend on the choice of v_0 or orientation of the loop.

Note in addition that the definition of the plaquette projectors in Eq. (7) uses F symbols and, in principle, these are defined only for processes with legal vertices, i.e., satisfying the fusion rules. One may set them to zero whenever one of the involved vertices is illegal, however this looks artificial. In Sec. IV we will see how the definition of B_p^{SN} for legal vertices agrees with and is naturally generalized by that of the quantum doubles for the subclass of string nets in the range of our mapping.

IV. MAPPING QUANTUM DOUBLES TO STRING NETS

Consider the D(G) model defined on a planar lattice Λ and perform a basis change at each oriented edge to the Fourier basis defined by

$$|\mu,a,b\rangle = \sqrt{\frac{|\mu|}{|G|}} \sum_{g\in G} [D^\mu(g)]_{ab}|g\rangle, \qquad (8)$$

where $\mu\in\hat{G}$ runs over the irreducible representations of G and D^μ is a fixed matrix realization of the irreducible representation μ (with dimension $|\mu|$.) Standard representation-theoretical orthogonality relations imply that $\tilde{\mathcal{B}}=\{|\mu,a,b\rangle\}$ is an orthonormal basis. The orientation-reversing isomorphism is given by $|\mu,a,b\rangle_e \mapsto |\mu^*,b,a\rangle_{e^*}$.

This change of basis can be interpreted loosely as splitting the local degrees of freedom into three subspaces, one labeled by the irreducible representations of G, the other two labeled by matrix elements of these representations (this is not a rigorous interpretation because the dimensions of the latter subspaces depend on the irreducible representation; the rigorous statement is the Peter-Weyl theorem.) We now argue that the matrix indices are naturally associated with the beginning and end of an oriented edge and that the effect of vertex projectors in Eq. (1) is to determine the contraction of these indices at each vertex so the degrees of freedom remaining after imposing vertex projectors are just the irreducible representations of G for which the model can be inter-

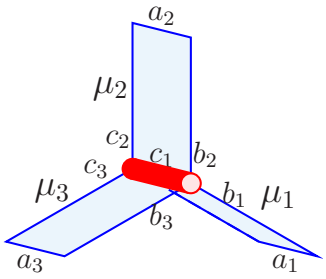

FIG. 5. (Color online) Graphical representation of the projector A_v^{QD} or more precisely its matrix element $\langle\{\mu_j,a_j,c_j\}|A_v^{\mathrm{QD}}|\{\mu_i,a_i,b_i\}\rangle$. Note that only the b indices change (into the c's), according to the corresponding $3j$ symbol $W^{\{\mu_i\}}_{\{c_i\},\{b_i\}}$; this is represented by the cylinder. The trivial propagation of the other indices is represented by the rectangular faces.

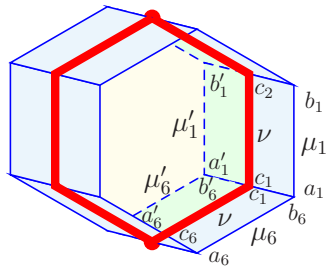

FIG. 6. (Color online) Graphical representation of the projector $B_p^{\mathrm{QD}}(\nu)$ or more precisely its matrix element $\langle\{\mu_j',a_j',b_j'\}|B_p^{\mathrm{QD}}(\nu)|\{\mu_j,a_j,b_j\}\rangle$ as given in Eq. (17). The string-net picture of a loop associated with irrep ν is represented by a transversal face (hexagon delimited by bold lines) that interacts with the propagation (rectangular faces) of the plaquette via $3j$ symbols (bold lines).

preted as a string-net model with fusion rules stemming from composition of irreducible representations.

Using the inverse change of basis

$$|g\rangle = \sqrt{\frac{|\mu|}{|G|}} \sum_{\mu\in\hat{G}} \sum_{a,b} [D^{\mu}(g)]^*_{ab} |\mu,a,b\rangle, \tag{9}$$

it is easy to check that

$$A_v^{\mathrm{QD}}|\{\mu_i,a_i,b_i\}\rangle_v = \sum_{c_1,\ldots,c_r} W^{\{\mu_i\}}_{\{c_i\},\{b_i\}} |\{\mu_i,a_i,c_i\}\rangle_v, \tag{10}$$

where

$$W^{\{\mu_i\}}_{\{c_i\},\{b_i\}} = \frac{1}{|G|} \sum_{\ell\in G} \prod_i [D^{\mu_i}(\ell)]_{c_i b_i} \tag{11}$$

is the projector onto the trivial isotypic subspace of $\otimes_i \mu_i$. In other words, it projects out vertex configurations in which the irreducible representations μ_i in the tensor product $\otimes_i \mu_i$ are not coupled to yield the trivial representation. This corresponds to the fusion rules in the string-net model to be identified below. A graphical interpretation of A_v^{QD} is given in Fig. 5.

Moreover, since $W^{\{\mu_i\}}$ is a projector, it can be split into a direct sum of orthogonal rank-one projectors, each one of which corresponds to an inequivalent fusion channel of the μ_i into the vacuum:

$$W^{\{\mu_i\}}_{\{a_i\},\{b_i\}} = \sum_A W^{\{\mu_i\},A}_{\{a_i\},\{b_i\}} = \sum_A w^{\{\mu_i\},A}_{\{a_i\}} (w^{\{\mu_i\},A}_{\{b_i\}})^*. \tag{12}$$

The number of such channels is the trace

$$\Delta_{\{\mu_i\}} = \mathrm{tr}\, W^{\{\mu_i\}}. \tag{13}$$

The action of A_v^{QD} fixes how the rightmost indices in the ket $|\{\mu_i,a_i,b_i\}\rangle_v$ should be contracted. Remember that we have defined this action assuming that all adjacent edges point toward v. Therefore, these indices correspond naturally to the ends of the oriented edges.

Now consider the action of the entire set of vertex projectors on the lattice. Then all matrix indices are contracted according to the annihilation channels of the incoming representations. Hence, if we consider the "physical" Hilbert space to be the surviving subspace after application of all A_v^{QD}, the only degrees of freedom left are precisely the irreducible representations of G living on oriented edges, with the constraint that representations incident on a given vertex can fuse to the vacuum.[19] We refer to the system in which just irrep labels are associated with oriented edges, obeying fusion rules, as the string-net lattice, and we identify a configuration $|\{\mu\}\rangle_{\mathrm{SN}}$ there with the state in the original model resulting from the appropriate contractions of matrix indices with eigenvectors of the vertex operators as defined in Eq. (12)

$$|\{\mu\}\rangle_{\mathrm{SN}} \mapsto \sum_{\{a,b\}} |\{\mu,a,b\}\rangle \prod_v (w_v)_{a\ldots b\ldots} \equiv |\{\mu\}\rangle_{\mathrm{QD}}. \tag{14}$$

In order to compute the action of the plaquette projectors B_p^{QD} in the Fourier basis, note that

$$\delta_{g,e} = \sum_{\nu\in\hat{G}} \frac{|\nu|}{|G|} \chi_\nu(g) \tag{15}$$

with $\chi_\nu = \mathrm{tr}\, D^\nu$ the character of the irreducible representation ν. Then

$$B_p^{\mathrm{QD}} = \sum_{\nu\in\hat{G}} \frac{|\nu|}{|G|} B_p^{\mathrm{QD}}(\nu) \tag{16}$$

with

$$\langle\{\mu_j',a_j',b_j'\}|B_p^{\mathrm{QD}}(\nu)|\{\mu_j,a_j,b_j\}\rangle = \prod_{i=1}^{s} \sqrt{|\mu_i||\mu_i'|} \sum_{c_1,\ldots,c_s} \prod_{i=1}^{s} W^{\mu_i\mu_i'^*\nu}_{a_i a_i' c_i, b_i b_i' c_{i+1}} \tag{17}$$

and the index i understood to be cyclic (see Fig. 6).

We assert that this is the correct action of plaquette projectors in the associated string-net model. Remember from[4] and Sec. III that this action is best understood in the fat lattice picture. Namely,

$$B_p^{\mathrm{SN}} = \sum_{\nu \in \hat{G}} \frac{|\nu|}{|G|} B_p^{\mathrm{SN}}(\nu), \tag{18}$$

where we have already identified the quantum dimensions of the labels as $d_\nu = |\nu|$ and the total quantum dimension as $\mathcal{D}^2 = \Sigma_\nu d_\nu^2 = |G|$. The operator $B_p^{\mathrm{SN}}(\nu)$ is equivalent to creating a loop of label ν around the puncture of plaquette p in the fat lattice and then subsuming it into the original lattice by means of F moves but in the case of group representations (and more general representation theories) F symbols are $6j$ symbols, which can be written entirely in terms of W projectors (thus, of $3j$ symbols) as defined in Eq. (11). This is explained in Appendix.

In order to identify the quantum double with a string-net model we restrict to a trivalent lattice, for definiteness of the honeycomb type. We need to show that the action of the $B_p^{\mathrm{QD}}(\nu)$ on the reduced QD states $|\cdots\rangle_{\mathrm{QD}}$ defined in Eq. (14) is the same as the action of $B_p^{\mathrm{SN}}(\nu)$ on states in the SN lattice. To this end we consider a hexagonal plaquette together with its external legs; in the string-net model its states are labeled by irreducible representations $\mu_1, \ldots, \mu_s$ for the edges in a counterclockwise loop along the plaquette boundary together with irrep labels $\alpha_1, \ldots, \alpha_s$ for the external legs oriented toward the plaquette. (The external leg with label α_j is supposed to end at the vertex where label μ_{j-1} enters and μ_j leaves.) Such a state we denote by $|\{\mu_j, \alpha_j\}\rangle_{\mathrm{SN}}$. The action of the string-net plaquette operators is:[4]

$$\begin{aligned}
&{}_{\mathrm{SN}}\langle\{\mu_j', \alpha_j\}|B_p^{\mathrm{SN}}(\nu)|\{\mu_j, \alpha_j\}\rangle_{\mathrm{SN}} \\
&= \prod_j F^{\alpha_j \mu_j'^{*} \mu_{j-1}'}_{\nu \mu_{j-1} \mu_j^{*}} \\
&= \prod_j \sqrt{|\mu_{j-1}'||\mu_j|} \sum_{a_j, m_j, \bar{m}_j, m_j', \bar{m}_j', n_j} (w^{\alpha_j \mu_j'^{*} \mu_{j-1}'}_{a_j m_j' \bar{m}_{j-1}'})^{*} \\
&\quad \times (w^{\nu \mu_{j-1} \mu_{j-1}'^{*}}_{n_j \bar{m}_{j-1} \bar{m}_{j-1}'})^{*} w^{\mu_{j-1} \alpha_j \mu_j^{*}}_{\bar{m}_{j-1} a_j m_j} w^{\mu_j'^{*} \nu \mu_j}_{m_j' n_j m_j},
\end{aligned} \tag{19}$$

where Eq. (A8) from Appendix has been used to express F symbols in terms of $3j$ symbols w.

In the quantum double model, we define states $|\{\mu_j, \alpha_j\}\rangle_{\mathrm{QD}}$ for the same system according to rule (14); the local Hilbert spaces are full group algebras labeled by irrep and matrix indices but the latter are contracted together with $3j$ symbols w. The action of the quantum double plaquette operator is

$$\begin{aligned}
&{}_{\mathrm{QD}}\langle\{\mu_j', \alpha_j\}|B_p^{\mathrm{QD}}(\nu)|\{\mu_j, \alpha_j\}\rangle_{\mathrm{QD}} \\
&= \prod_j \sum_{a_j, m_j, \bar{m}_j} \sum_{a_j', m_j', \bar{m}_j'} (w^{\mu_j'^{*} \mu_{j-1}' \alpha_j)})^{*} w^{\mu_j^{*} \mu_{j-1} \alpha_j}_{m_j \bar{m}_{j-1} a_j} \\
&\quad \times \langle\{\mu_j', m_j', \bar{m}_j'\}, \{\alpha_j, b_j, a_j'\}| \\
&\quad \times B_p^{\mathrm{QD}}(\nu)|\{\mu_j, m_j, \bar{m}_j\}, \{\alpha_j, b_j, a_j\}\rangle.
\end{aligned} \tag{20}$$

Note that the result is independent of the b_j chosen. In the right-hand side of the last equation $B_p^{\mathrm{QD}}(\nu)$ only acts on the edges of the plaquette according to Eq. (17), hence

$$\begin{aligned}
&{}_{\mathrm{QD}}\langle\{\mu_j', \alpha_j\}|B_p^{\mathrm{QD}}(\nu)|\{\mu_j, \alpha_j\}\rangle_{\mathrm{QD}} \\
&= \prod_j \sqrt{|\mu_j||\mu_j'|} \sum_{a_j, m_j, \bar{m}_j, m_j', \bar{m}_j', c_j} (w^{\mu_j'^{*} \mu_{j-1}' \alpha_j})^{*} \\
&\quad \times w^{\mu_j^{*} \mu_{j-1} \alpha_j}_{m_j \bar{m}_{j-1} a_j} W^{\mu_j \mu_j'^{*} \nu}_{m_j m_j' c_j, \bar{m}_j \bar{m}_j' c_{j+1}},
\end{aligned} \tag{21}$$

which coincides with Eq. (19), that is,

$$\begin{aligned}
&{}_{\mathrm{QD}}\langle\{\mu_j', \alpha_j\}|B_p^{\mathrm{QD}}(\nu)|\{\mu_j, \alpha_j\}\rangle_{\mathrm{QD}} \\
&= {}_{\mathrm{SN}}\langle\{\mu_j', \alpha_j\}|B_p^{\mathrm{SN}}(\nu)|\{\mu_j, \alpha_j\}\rangle_{\mathrm{SN}}.
\end{aligned} \tag{22}$$

Let us comment on the structure of this mapping. The SN definition of plaquette operators relies on F symbols, whose extension to configurations violating vertex conditions is somewhat arbitrary. By enlarging the SN local Hilbert spaces introducing matrix degrees of freedom and going over to the QD Hilbert spaces where edges carry a full group algebra, we are able to express both plaquette and vertex operators in a way that recovers the SN definition for the reduced states defined in Eq. (14) but carries over to the full Hilbert space. In more concrete terms, we can write

$$B_p^{\mathrm{SN}} \sim B_p^{\mathrm{QD}} \otimes \bigotimes_{v \text{ around } p} A_v^{\mathrm{QD}} \tag{23}$$

in the sense that B_p^{SN} needs the vertices surrounding the plaquette to fulfill the fusion rules and in that space its action can be identified with that of B_p^{SN}; incidentally, this accounts for the fact that the SN plaquette operators are 12-local while the QD plaquette operators are 6-local.

String-net models obtained from quantum doubles by the Fourier mapping can be defined naturally for general planar lattices and not only in trivalent lattices as a generic SN model. The reason is that the vertex projectors have a natural interpretation in group-representation theory, which generalizes to n-valent vertices: a vertex configuration is allowed if the tensor product of the incident irreducible representations contains the trivial representation.

Moreover, group theory also provides us with a natural splitting of the F symbols according to Eq. (A8) in Appendix, implying that plaquette projectors act effectively only on the edges of the plaquettes since the parts associated with the external legs have the form of vertex projectors and act trivially on physical states.

More generally, we have an identification of the superselection sectors as irreducible representations of (in this case) the quasitriangular Hopf algebra $\mathrm{D}(G)$. Note that the matrix degrees of freedom a and b which must be added to the string-net lattice to fill the quantum double Hilbert spaces with basis $\{|\mu a b\rangle\}$ allow us to keep track of the internal degrees of freedom within the different irreps of $\mathrm{D}(G)$ (e.g., the group element labels for the conjugacy classes defining the magnetic fluxes, the different vectors for the irreps of the group in electric charges).

From a more abstract point of view, both quantum doubles and their corresponding string-net models can be seen as a procedure to obtain an anyon model, that of the physical excitations, which is a unitary braided tensor category. This has as objects the superselection sectors, i.e., the

excitations classified by irreducible representations of D(G). This is both obtained starting with the model defined à la QD, i.e., starting with a basis labeled by group elements (objects of a category G) and with the model defined à la SN with bases labeled by irreps (objects of a category of representations of G). These categories are equivalent in the sense that they have the same excitations.

V. CONCLUSIONS AND OUTLOOK

We have shown explicitly how to identify Kitaev's quantum double models[3] with a subclass of the string-net models of Levin and Wen's.[4] The general construction for string nets can be further simplified in this case due to the interpretation of the fusion rules in terms of group theory.

As a result, the subclass of SN models corresponding to QD models can be extended naturally to arbitrary planar lattices; their excitations can be given a representation-theoretic interpretation at the price of introducing auxiliary degrees of freedom necessary to keep track of the internal spaces of the different representations; and the electric-magnetic duality is recovered, in that plaquette projectors can be given a natural definition that does not depend on the completion of F symbols outside the space of recouplings with legal vertices. This provides a local characterization of excitations which we find satisfactory.

Interestingly, from the point of view of category theory the construction can be seen as an instance of Morita equivalence, which stresses the practical importance of these models as laboratories to provide simple examples of abstract mathematical notions which, in spite of their importance, are only in their way to become everyday tools of theoretical physicists.

Let us stress the significance of this construction. On the one hand, it is a nontrivial mapping relating the physics of two different classes of topological models. We have tried to emphasize the interplay of physical degrees of freedom which is needed to show this relationship and how the smaller local Hilbert space for the string-net lattice can be naturally enlarged to the local Hilbert space of the corresponding quantum double model. On the other hand, it allows for a clearer picture of the anyons appearing as physical excitations of the particular class of string nets obtained from our mapping and this picture can be extended to more general string-net models as we mention below.

This construction will help throw light as well on the relations among different tensor network constructions developed recently to describe exactly both quantum doubles[17,20] and string-net models.[18,21,22] On the other hand, it is necessary to discuss the relation between Kitaev's ribbon operators[3] and the loop operators defined by Levin and Wen.[4]

The current construction can be extended to models based on local degrees of freedom where the Hilbert space constitutes a C^*-Hopf algebra (as anticipated in Ref. 3); this generalization will be given in Ref. 23, where the corresponding relations among tensor network descriptions will also be discussed. The case of models based on weak quasi-Hopf algebras[24] is the subject of work in progress.[25] This will extend the representation-theoretic approach to excitations to wider subclasses of string-net models.

Note added. Recently, related work (including a discussion of ribbon operators) was reported in Ref. 26.

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APPENDIX: EXPRESSION OF F SYMBOLS IN TERMS OF $3j$ SYMBOLS

It can be shown that the projector onto the trivial representation subspace of the product $\mu\otimes\nu\otimes\lambda\otimes\rho$ of irreducible representations of group G splits into a sum of orthogonal projectors associated with the internal channel σ in the coupling scheme $\mu\otimes\nu\overset{\sigma}{\longrightarrow}\lambda\otimes\rho$, say, as

$$W^{\mu\nu\lambda\rho}=\bigoplus_{\sigma\in\hat{G}}\Pi_{\sigma}^{\mu\nu,\lambda\rho}, \tag{A1}$$

where the projectors

$$(\Pi_{\sigma}^{\mu\nu,\lambda\rho})_{mn\ell r,\bar{m}\bar{n}\bar{\ell}\bar{r}}=|\sigma|\sum_{s,\bar{s}}W^{\mu\nu\sigma}_{mns,\bar{m}\bar{n}\bar{s}}W^{\lambda\rho\sigma^*}_{\ell rs,\bar{\ell}\bar{r}\bar{s}} \tag{A2}$$

are expressed in terms of W connecting three irreducible representations.[27] This leads to the definition of the $\hat{F}$ operation as the change of basis within the range of $W^{\mu\nu\lambda\rho}$, from the states associated with $\Pi_{\sigma}^{\mu\nu,\lambda\rho}$ to those of the alternative coupling scheme $\Pi_{\tau}^{\rho\mu,\nu\lambda}$. Explicitly, the operators read

$$\hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau}=\Pi_{\sigma}^{\mu\nu,\lambda\rho}\Pi_{\tau}^{\rho\mu,\nu\lambda} \tag{A3}$$

and obviously commute with $W^{\mu\nu\lambda\rho}$. From here it is immediate to check, for instance, that

$$\sum_{\tau}\hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau}\hat{F}^{\rho\mu\tau}_{\nu\lambda\xi}=\delta_{\sigma\xi^*}\Pi_{\sigma}^{\mu\nu,\lambda\rho}. \tag{A4}$$

In components, taking into account Eq. (12) for rank-one projectors, one has

$$(\hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau})_{mn\ell r,\bar{m}\bar{n}\bar{\ell}\bar{r}}=(p_{\sigma}^{\mu\nu,\lambda\rho})_{rmn\ell}F^{\mu\nu\sigma}_{\lambda\rho\tau}(p_{\tau}^{\rho\mu,\nu\lambda})^*_{\bar{m}\bar{n}\bar{\ell}\bar{r}}, \tag{A5}$$

where

$$(p_{\sigma}^{\mu\nu,\lambda\rho})_{mn\ell r}=\sqrt{|\sigma|}\sum_{s}w^{\mu\nu\sigma}_{mns}w^{\lambda\rho\sigma^*}_{\ell rs} \tag{A6}$$

are +1 eigenvectors of the (rank-one) projectors in Eq. (A2)

$$(\Pi_{\sigma}^{\mu\nu,\lambda\rho})_{mn\ell r,\bar{m}\bar{n}\bar{\ell}\bar{r}}=(p_{\sigma}^{\mu\nu,\lambda\rho})_{mn\ell r}(p_{\sigma}^{\mu\nu,\lambda\rho})^*_{\bar{m}\bar{n}\bar{\ell}\bar{r}} \tag{A7}$$

and

$$F^{\mu\nu\sigma}_{\lambda\rho\tau}=\sqrt{|\sigma||\tau|}\sum_{mn\ell rst}(w^{\mu\nu\sigma}_{mns})^*(w^{\lambda\rho\sigma^*}_{\ell rs})^*w^{\rho\mu\tau}_{rmt}w^{\nu\lambda\tau^*}_{n\ell t}, \tag{A8}$$

for which, for instance,

$$\sum_{\tau}F^{\mu\nu\sigma}_{\lambda\rho\tau}F^{\rho\mu\tau}_{\nu\lambda\xi}=\Delta_{\mu\nu\sigma}\Delta_{\lambda\rho\xi}\delta_{\sigma\xi^*} \tag{A9}$$

and

$$F^{\mu\nu 0}_{\lambda\rho\tau} = \sqrt{\frac{|\tau|}{|\mu||\lambda|}}\delta_{\mu\nu^*}\delta_{\lambda\rho^*}\Delta_{\mu\lambda^*\tau} \tag{A10}$$

(up to a phase from the square root).

Now the effect of the $\hat{F}$ operators can be interpreted directly in the string-net lattice by forgetting about the p tensors whose role is enforcing physical constraints throughout. The $F^{\mu\nu\sigma}_{\lambda\rho\tau}$ are the same as the F symbols in Ref. 4.

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